Super-additivity of quantum-correlating power

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We prove that, when two local quantum channels are used paralleled, the quantum-correlating power (QCP) of the composed channel is no less than the sum of QCP of the two channels. For local channels with zero QCP, the super-activation of QCP is a fairly common effect, and proved to exist except for the trivial case where both of the channels are completely decohering channels or unitary operators. For general quantum channels, we show that the (not-so-common) additivity of QCP can be observed for the situation where a measuring-and-preparing channel is used together with a completely decohering channel.

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I. INTRODUCTION

Contrary to quantum entanglement, which is a monotone under local operations, general quantum correlations, such as quantum discord, can be created and increased by local operations [1–5]. This is comprehensive, as the local operations can turn some classical correlation into quantum one. More and more evidences show that quantum correlation is responsible for quantum information processes, such as remote state preparation [6, 7], entanglement distribution [8, 9], quantum state discrimination [10], etc. The importance of quantum correlation also lies in its close connection to quantum entanglement [11–13]. The local creation of quantum correlation makes the preparation of a quantum correlated state simple. Meanwhile, it provides a method to study the properties of both the quantum correlations and local quantum channels. It has attracted much attention recently. Generally, a local channel is able to create quantum correlation if and only if it is not a commutativity-preserving channel [2]. For qubit case, a commutativity-preserving channel is either a completely decohering channel or a unital channel [1, 2], while for qutrit case, the set of commutativity-preserving channels reduces to completely decohering channels and isotropic channels [2].

On solving the problem of whether a channel can create quantum correlation, it is natural to investigate how much quantum correlation can be created locally. For this purpose, QCP was proposed as the maximum quantum correlation that can be created by a given local quantum channel [14]. QCP not only quantifies the amount of quantum correlation created by local operation, but also serves as an inherent property of quantum channels. It is of interest to investigate the effect caused by using two channels together. We have given an example to indicate the super-activation for QCP of two zero-QCP channels in Ref. [14]. It is straight forward to ask the following questions: What kind of local channels has the property of super-activation of QCP? Is super-additivity of QCP holds for general quantum channels?

In this paper, we prove that when the two channels have zero-QCP, the super-activation can be observed except that the two channels are both unitary operations or completely decohering channels. It means that the super-activation of QCP is a common phenomenon. We then prove that the answer to the first question is positive. When one of the channels have positive QCP, there are still situations where the QCP of the two channels are additive. The QCP of the bi-channel which is composed of a measuring-and-preparing (MP) channel and a completely decohering channel equals to that of the MP channel. We consider the genuine quantum correlation to be responsible for the super-additivity of QCP.

II. SUPER-ACTIVATION OF QUANTUM-CORRELATING POWER

Let us briefly recall the definition of QCP [14] for local channels. The quantum-correlating power of a quantum channel Λ is defined as

$$\mathcal{Q}(\Lambda) = \max_{\rho\in\mathcal{C}_0} Q(\Lambda\otimes I(\rho)). \tag{1}$$

Here $\mathcal{C}_0$ is a set of classical-quantum states, which can be written as [15]

$$\mathcal{C}_0 = \{\rho|\rho = \sum_i q_i \Pi^A_{\alpha_i}\otimes\rho^B_i\}, \tag{2}$$

and Q is a measure of quantum correlation satisfying the following three conditions: (a) $Q(\rho) = 0$ iff $\rho\in\mathcal{C}_0$; (b) $Q(U\rho U^\dagger) = Q(\rho)$ where U is a local unitary operator on A or B; (c) $Q(I\otimes\Lambda_B(\rho)) \le Q(\rho)$.

In Ref. [14], we have given an example to show the super-activation of QCP. When two phase-damping qubit channels $\Lambda^{\rm PD}$ are used paralleled, the QCP of the composed channel $\Lambda^{\rm PD}\otimes\Lambda^{\rm PD}$ is nonzero. Precisely, there is a classical-quantum state $\rho^{AA'\tilde{B}}$ with qubits A and

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A' on the left and system $\tilde{B}$ on the right, such that $Q(\Lambda_A^{\mathrm{PD}} \otimes \Lambda_{A'}^{\mathrm{PD}}(\rho_{AA'\tilde{B}})) \neq 0$. Then what kind of zero-QCP channels has the property of super-activation of QCP? Obviously, if the two channels are both completely decohering channels, the composed channel is still a completely decohering channel, which is not able to create quantum correlation. Similarly, for the case that the two channels are both unitary operators, the composed channel is a unitary operator too, and thus has zero-QCP. In the following, we will prove that these are the only two situations where super-activation of QCP is not observed.

Theorem 1. For two zero-QCP channels Λ_1 and Λ_2, the QCP of the composite channel $\Lambda_1 \otimes \Lambda_2$ is non-zero except that both Λ_1 and Λ_2 are completely decohering channels or the unitary channels.

Proof. From Ref. [2], the QCP of channel $\Lambda_1 \otimes \Lambda_2$ is zero if and only if $\Lambda_1 \otimes \Lambda_2$ is a commutativity-preserving channel. It means that, for any commutative two-particle states ξ_1 and ξ_2, we have

$$[\Lambda_1^A \otimes \Lambda_2^{A'}(\xi_1), \Lambda_1^A \otimes \Lambda_2^{A'}(\xi_2)] = 0. \tag{3}$$

Obviously, when both Λ_1 and Λ_2 are completely decohering channels or unitary channels, the channel $\Lambda_1 \otimes \Lambda_2$ is also a commutativity-preserving channel, Eq. (3) holds. Otherwise, let Λ_1 not be a completely decohering channel and Λ_2 not be a unitary channel. Now we choose $\xi_1 = |0\rangle_A\langle 0| \otimes |\psi\rangle_{A'}\langle\psi|$ and $\xi_2 = |\theta\rangle_A\langle\theta| \otimes |\psi^\perp\rangle_{A'}\langle\psi^\perp|$, where $\langle\psi|\psi^\perp\rangle = 0$ and $|\theta\rangle$ is an arbitrary single-particle state. Then Eq. (3) is equivalent to

$$[\Lambda_1(|0\rangle\langle 0|), \Lambda_1(\theta)] \otimes \Lambda_2(\psi)\Lambda_2(\psi^\perp) = 0. \tag{4}$$

Here we label ψ as the density matrix as the pure state $|\psi\rangle$ and similar for $\psi^\perp$ and θ. Since channel Λ_1 is not a completely decohering channel, we can always find a single-particle state $|\theta\rangle$ such that $[\Lambda_1(|0\rangle\langle 0|), \Lambda_1(\theta)] \neq 0$. Meanwhile, when Λ_2 is not a unitary channel, there exist two pure orthogonal single-particle states ψ and $\psi^\perp$ such that $\Lambda_2(\psi)\Lambda_2(\psi^\perp) \neq 0$. Therefore, Eq. (4) is violated. It completes the proof of Theorem 1.

In the above discussion, super-activation of QCP is due to the non-classicality of particle A. An extreme example is that, $\Lambda_1 = I$ is an identity channel while Λ_2 is a completely depolarizing channel, which is equivalent to the "trace-out" operation. Now we start with the state $\rho_{AB} \otimes |0\rangle_{A'}\langle 0|$, where $\rho_{AB} \in \mathcal{C}_0$. After a two-particle unitary operator $U_{AA'}$ on A and A', which does not create quantum correlation on the left, the channel $I \otimes \Lambda_2$ is applied. This process can be expressed as

$$\rho_{\mathrm{out}} = \Lambda_A^U(\rho_{AB}) \otimes \frac{I_{A'}}{2}. \tag{5}$$

where $\Lambda_A^U(\rho_A B) = \mathrm{Tr}_{A'}(U_{AA'}\rho_{AB} \otimes |0\rangle_{A'}\langle 0|U_{AA'}^\dagger)$. Therefore, the super-activation of the two channels $\Lambda_1 = I$ and Λ_2 is in fact local creation of quantum correlation by the channel Λ_A^U.

We will then focus on situations where no pairwise quantum correlations are induced between A and $\tilde{B}$ or between A' and $\tilde{B}$. The mechanism for super-activation of QCP under this condition is totally different from that for local creation of quantum correlation. In this case, the two states ξ_1 and ξ_2 used for checking nonzero QCP of the channel $\Lambda_1 \otimes \Lambda_2$ should satisfy

$$[\xi_1^A, \xi_2^A] = 0,\ [\xi_1^{A'}, \xi_2^{A'}] = 0. \tag{6}$$

Obviously, when channel Λ_2 is a completely decohering channel, the left hand side of Eq. (3) equals to $[\Lambda_1(\xi_1^A), \Lambda_1(\xi_2^A)] = 0$ under the constraint of Eq. (6) and the super-activation of QCP can not be observed. In the following, we will see that, even limited to the situation where no pairwise quantum correlation is induced, the super-activation of QCP can still be observed for most of the channels.

We first discuss the situation where Λ_1 and Λ_2 are the qubit channels. According to [1, 2], Λ has zero-QCP only when it is a completely decohering channel Λ^{CD} or a unital channel Λ^{I}. Since $\Lambda^{\mathrm{CD}} \otimes \Lambda^{\mathrm{CD}}$ has zero-QCP, we will focus on the super-activation of QCP for two unital channels. A unital channel is defined as a channel which keeps the identity operator invariant

$$\Lambda^{\mathrm{I}} \equiv \{\Lambda : \Lambda(I) = I\}. \tag{7}$$

It has been proved that any unital channel of a qubit is unitarily equivalent to a Pauli channel [16] $\Lambda^{\mathrm{I}}(\cdot) = v\Lambda^{\mathrm{Pauli}}(u^\dagger(\cdot)u)v^\dagger$. Here the Kraus operators of a Pauli channel are proportional to Pauli matrices

$$\Lambda^{\mathrm{Pauli}}(\cdot) = \sum_{i=0}^{3} \lambda_i \sigma_i(\cdot)\sigma_i, \tag{8}$$

where $\sigma_0 = I$, $\sigma_{1,2,3}$ are the three Paule matrices, $\lambda_i \geq 0$, $\lambda_0 \geq \lambda_{1,2,3}$, and $\sum_{i=0}^{3} \lambda_i = 1$. Therefore, it is adequate to consider the super-activation of QCP for Pauli channels.

Theorem 2. When limited to the case where no pairwise quantum correlation is induced, the super-activation of QCP can be observed for unital qubit channels which are not one of the following cases: (a) both channels are identical isotropic channels, and (b) one of the channels is a completely depolarizing channel.

Proof. Case (b) is obvious from the above discussions, so we focus on the cases that none of the two channels are completely depolarizing channels. ξ_1 and ξ_2 in Eq. (3) is chosen as $\xi_i = |\Phi_i\rangle\langle\Phi_i|$, where $|\Phi_1\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$ and $|\Phi_2\rangle = (|0+\rangle + |1-\rangle)/\sqrt{2}$. In Pauli presentation, we have

$$\begin{aligned}
\xi_1 &= \frac{1}{4}(I \otimes I + \sigma_1 \otimes \sigma_1 - \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_3) \\
\xi_2 &= \frac{1}{4}(I \otimes I + \sigma_1 \otimes \sigma_3 + \sigma_2 \otimes \sigma_2 + \sigma_3 \otimes \sigma_1).
\end{aligned} \tag{9}$$

For Λ_1 being a Pauli channel, we have $\Lambda_1(\sigma_i) = a_i\sigma_i$ with $a_i = \lambda_0^{(1)} + 2\lambda_i^{(1)} - \sum_i \lambda_i^{(1)}$, $i = 1, 2, 3$. Similarly, $\Lambda_2(\sigma_i) = b_i\sigma_i$. Consequently, $\Lambda_1 \otimes \Lambda_2(\xi_1) = \frac{1}{4}(I \otimes I + a_1b_1\sigma_1 \otimes \sigma_1 - a_2b_2\sigma_2 \otimes \sigma_2 + a_3b_3\sigma_3 \otimes \sigma_3)$ and $\Lambda \otimes \Lambda(\xi_2) = \frac{1}{4}(I \otimes$

$I + a_1 b_3 \sigma_1 \otimes \sigma_3 + a_2 b_2 \sigma_2 \otimes \sigma_2 + a_3 b_1 \sigma_3 \otimes \sigma_1)$. Therefore, Eq. (3) means that

$$a_1 a_3 (b_1^2 - b_3^2) = 0, b_1 b_3 (a_1^2 - a_3^2) = 0. \quad (10)$$

By alternating the subscript 1, 2, 3 in Eq. (9) and adopting Eq. (3), we have

$$\begin{aligned} a_2 a_3 (b_2^2 - b_3^2) = 0, b_2 b_3 (a_2^2 - a_3^2) = 0, \\ a_1 a_2 (b_1^2 - b_2^2) = 0, b_1 b_2 (a_1^2 - a_2^2) = 0. \end{aligned} \quad (11)$$

It means that the super-activation of QCP for channels violating Eqs. (10) or (11) can be detected by the states ξ_1 and ξ_2 in form of Eq. (9) or those obtained by alternating the subscript 1, 2, 3 in Eq. (9). Then we are left with the following situations.

(1) $|a_2| = |a_3| = a > 0$, $|b_2| = |b_3| = b > 0$, and $a_1 = b_1 = 0$. Noticing that conditions $a_1 = 0$ and $\lambda_0 \geq \lambda_{1,2,3}$ lead to $a_{2,3} \geq 0$, we have $a_2 = a_3 = a > 0$ and $b_2 = b_3 = b > 0$. Λ_1 and Λ_2 are both projecting-and-depolarizing channels Λ^{Dp}, which takes a state

$$\zeta = (I + \vec{r} \cdot \vec{\sigma})/2, \quad (12)$$

to $\Lambda_\chi^{\mathrm{Dp}}(\zeta) = [I + \chi(r_2\sigma_2 + r_3\sigma_3)]/2$, where $\chi = a, b$. In other words, the channel projects the Bloch vector $\vec{r}$ onto $x - y$ plain and then shorten it, $\Lambda_\chi^{\mathrm{Dp}} \equiv \Lambda_\chi^{\mathrm{D}} \circ \Lambda^p$.

(2) $|a_2| = |a_3| = a > 0$, $b_2 = b_3 = a_1 = 0$, and $b_1 \neq 0$. Similar discussions as in Case (1) give that $a_2 = a_3 = a > 0$ and $b_1 = b > 0$. $\Lambda_1 = \Lambda_a^{\mathrm{Dp}}$ while $\Lambda_2 = \Lambda_b^{\mathrm{DPD}} \equiv \Lambda_b^{\mathrm{D}} \circ \Lambda^{\mathrm{PD}}$ is equivalent to a completely dephasing channel Λ^{PD} followed by a depolarizing channel Λ_b^{D}.

(3) $|a_1| = |a_2| = |a_3| > 0$ and $|b_1| = |b_2| = |b_3| > 0$. If $a_1 = -a_2 = a_3 \neq 0$, we have $\lambda_0 = \lambda_1 = \lambda_3 > \lambda_2$. This channel is equivalent to the isotropic channel with $\lambda_1 = \lambda_2 = \lambda_3 > \lambda_0$ by a unitary operator σ_2. Therefore, this case is equivalent to $a_1 = a_2 = a_3 = a \neq 0$ and $b_1 = b_2 = b_3 = b \neq 0$. Λ_1 and Λ_2 are isotropic channels.

(4) $b_1 = b_2 = b_3 = 0$. Channel Λ_2 is a completely depolarizing channel, which we have already considered.

Since two completely decohering channels do not have the property of super-activation of QCP, let Λ_1 not be a completely decohering channel. Cases (1) and (2) include the channels obtained by alternating the subscripts 1,2,3. In the following, we will derive the commutative states ξ_1 and ξ_2 satisfying Eq. (6) to detect the super-activation for these cases, and prove that such states does not exist for Case (3) with $a_1 = b_1$.

Writing a two-qubit state in form

$$\xi_k = \frac{1}{4}[I \otimes I + \sum_{i=1}^{3} r_i^k \sigma_i \otimes I + \sum_{i=1}^{3} s_i^k I \otimes \sigma_i + \sum_{i,j=0}^{3} T_{ij}^k \sigma_i \otimes \sigma_j], \quad (13)$$

where $k = 1, 2$, we can present ξ_k as $\xi_k = \{\vec{r}^k, \vec{s}^k, \hat{T}^k\}$. It is worth noting that the commutation $[\xi_1, \xi_2]$ can also be written as the Bloch decomposition

$$[\xi_1, \xi_2] = \frac{\tilde{i}}{16}[\sum_{i=1}^{3} \alpha_i \sigma_i \otimes I + \sum_{i=1}^{3} \beta_i I \otimes \sigma_i + \sum_{i,j=0}^{3} \Gamma_{ij} \sigma_i \otimes \sigma_j], \quad (14)$$

where $\tilde{i}$ is the imaginary unit. By using the commutation for Pauli matrices, we have $[\sigma_i \otimes I, \sigma_{i'} \otimes \sigma_{j'}] = [\sigma_i, \sigma_{i'}] \otimes \sigma_{j'}$, $[\sigma_i \otimes I, \sigma_i' \otimes I] = [\sigma_i, \sigma_{i'}] \otimes I$, and $[\sigma_i \otimes \sigma_j, \sigma_{i'} \otimes \sigma_{j'}] = [\sigma_i, \sigma_{i'}] \otimes I \delta_{jj'} + I \otimes [\sigma_j, \sigma_{j'}] \delta_{ii'}$, where $\delta_{jj'}$ is the Kroneker delta. Therefore,

$$\begin{aligned} \Gamma_{ij} &= (\vec{r}^1 \times \vec{T}_{r_j}^2 - \vec{r}^2 \times \vec{T}_{r_j}^1)_i + (\vec{s}^1 \times \vec{T}_{s_i}^2 - \vec{s}^2 \times \vec{T}_{s_i}^1)_j, \\ \vec{\alpha} &= \vec{r}^1 \times \vec{r}^2 + \sum_j \vec{T}_{r_j}^1 \times \vec{T}_{r_j}^2, \\ \vec{\beta} &= \vec{s}^1 \times \vec{s}^2 + \sum_i \vec{T}_{s_i}^1 \times \vec{T}_{s_i}^2, \end{aligned} \quad (15)$$

where $\vec{T}_{r_j}^k = \{T_{1j}^k, T_{2j}^k, T_{3j}^k\}$ and $\vec{T}_{s_i}^k = \{T_{i1}^k, T_{i2}^k, T_{i3}^k\}$. Then $[\xi_1, \xi_2] = 0$ is equivalent to

$$\vec{\alpha} = \vec{\beta} = 0, \hat{\Gamma} = 0. \quad (16)$$

Meanwhile, $[\xi_1^A, \xi_2^A] = 0$ and $[\xi_1^{A'}, \xi_2^{A'}] = 0$ give that

$$\vec{r}^1 \times \vec{r}^2 = 0, \vec{s}^1 \times \vec{s}^2 = 0. \quad (17)$$

For Cases (1) and (2), as well as Case (3) with $a \neq b$, we can find states ξ_1 and ξ_2 satisfying Eqs. (16) and (17) such that Eq. (3) is violated. For example, for Case (1) we chose ξ_k to be

$$\begin{aligned} \vec{r}^1 &= \vec{s}^1 = \{0, 0, r\}, \vec{r}^2 = \vec{s}^2 = \{0, 0, nr\}, \\ \hat{T}^1 &= \hat{T}^2 = \mathrm{diag}\{t, t, t\}. \end{aligned} \quad (18)$$

Direct calculation leads to $[\Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{Dp}}(\xi_1), \Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{Dp}}(\xi_2)] = \tilde{i}abrt(1-n)(b\sigma_2 \otimes \sigma_1 + a\sigma_1 \otimes \sigma_2) \neq 0$. For Case (2), we chose ξ_1 and ξ_2 to be

$$\begin{aligned} \vec{s}^1 &= \vec{r}^2 = \{0, r, r\}, \vec{r}^1 = \vec{s}^2 = 0, \\ \hat{T}^1 &= (\hat{T}^2)^{\mathrm{T}} = \{0, 0, 0; t, 0, 0; -t, 0, 0\}, \end{aligned} \quad (19)$$

and then $[\Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{DPD}}(\xi_1), \Lambda_a^{\mathrm{Dp}} \otimes \Lambda_b^{\mathrm{DPD}}(\xi_2)] = 2\tilde{i}abrt\sigma_1 \otimes \sigma_1 \neq 0$. For Case (3) with $a \neq b$, we chose ξ_1 and ξ_2 as in Eq. (19), and then $[\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_b^{\mathrm{Iso}}(\xi_1), \Lambda_a^{\mathrm{Iso}} \otimes \Lambda_b^{\mathrm{Iso}}(\xi_2)] = 2\tilde{i}abrt(a-b)\sigma_1 \otimes \sigma_1 \neq 0$.

Now we only need to prove that for two identical isotropic channels, the super-activation of QCP can not be detected when under the constraint that no pairwise quantum correlation is induced. From the property of the isotropic channels

$$\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_k) = \{a\vec{r}^k, a\vec{s}^k, a^2\hat{T}^k\}, \quad (20)$$

and consequently, the commutation of the output states can be written as $[\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_1), \Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_2)] = \{\vec{\alpha}^{\mathrm{Iso}}, \vec{\beta}^{\mathrm{Iso}}, \hat{\Gamma}^{\mathrm{Iso}}\}$, where $\vec{\alpha}^{\mathrm{Iso}} = a^2 \vec{r}^1 \times \vec{r}^2 + a^4 \sum_j \vec{T}_{r_j}^1 \times \vec{T}_{r_j}^2$, $\vec{\beta}^{\mathrm{Iso}} = a^2 \vec{s}^1 \times \vec{s}^2 + a^4 \sum_i \vec{T}_{s_i}^1 \times \vec{T}_{s_i}^2$, and $\hat{\Gamma}^{\mathrm{Iso}} = a^3 \hat{\Gamma}$. Therefore, Eqs. (16) and (17) imply $\vec{\alpha}^{\mathrm{Iso}} = \vec{\beta}^{\mathrm{Iso}} = 0$ and $\hat{\Gamma}^{\mathrm{Iso}} = 0$. It means that for two identical isotropic channels, the super-activation of QCP can not be detected under the constraint that no pairwise quantum correlation is induced. This completes the proof of theorem 2.

It is quite interesting that, a completely dephasing channel can activate the QCP of a depolarizing channel, even under the constraint that no pairwise quantum correlation is induced. More precisely, we consider initial classical-classical state $\rho = \sum_{i=0}^{3} |\Phi_i\rangle_{AA'}\langle\Phi_i| \otimes |i\rangle_B\langle i|$, where $|\Phi_0\rangle = (|01\rangle - |10\rangle)/\sqrt{2}$, $|\Phi_3\rangle = (|-0\rangle - |+1\rangle)/\sqrt{2}$, and $|\Phi_{1,2}\rangle$ are the same as in Eq. (9). Let Λ_A be a depolarizing channel with $\lambda_0 = (1+3a)/4$ and $\lambda_1 = \lambda_2 = \lambda_3 = (1-a)/4$ in Eq. (8), and $\Lambda_{A'}$ be a completely dephasing channel with $\lambda_0 = \lambda_3 = 1/2$ and $\lambda_1 = \lambda_2 = 0$. The output state $\Lambda_A \otimes \Lambda_{A'} \otimes I_B(\rho)$ has zero quantum correlation since $[\Lambda_A \otimes \Lambda_{A'}(\Phi_1), \Lambda_A \otimes \Lambda_{A'}(\Phi_2)] = -\frac{\tilde{i}}{4}a^2\sigma_2 \otimes \sigma_0$, which is nonzero for $a \neq 0$. Let us look more closely at the correlation structure of the input state ρ. Clearly, the pairwise quantum correlation between any two particles is zero. However, the genuine quantum correlation is nonzero. This is because the entanglement between the qubit A and the composite system $A'B$. Measuring B on basis $\{|i\rangle\}$ and locally operating A and A' can distill a singlet of qubits A and A'. Therefore, the super-activation of QCP can be understood as transfering the genuine quantum correlation to the quantum correlation between the bipartition AA' and B. Notice that the generated quantum correlation is still genuine correlation, because the pairwise quantum correlation is still zero in the output state.

An equivalent statement for Theorem 2(a) is that for two commutative two-qubit states ξ_1 and ξ_2 which satisfy Eq. (6), the following equation holds

$$[\xi_1, \xi_2^A + \xi_2^{A'}] = [\xi_2, \xi_1^A + \xi_1^{A'}]. \tag{21}$$

The equivalence is obvious when we notice that $\Lambda_a^{\mathrm{Iso}} \otimes \Lambda_a^{\mathrm{Iso}}(\xi_k) = a^2\xi_k + a(1-a)(\xi_k^A + \xi_k^{A'})/2 + I^{AA'}/4$. However, Eq. (21) does not always hold for A and A' being qu$d$its with $d \geq 3$. For example, consider $d = 4$ and that the two commutative two-qu$d$it states are $\xi_1 = [I + T_1(\sigma_1 \otimes \sigma_2)_A \otimes (\sigma_3 \otimes \sigma_1)_{A'} + T_2(\sigma_3 \otimes \sigma_1)_A \otimes (\sigma_3 \otimes \sigma_2)_{A'}]/16$ and $\xi_2 = [I + T_1 I^A \otimes (\sigma_1 \otimes \sigma_2)_{A'} + T_2(\sigma_2 \otimes \sigma_3)_A \otimes (\sigma_1 \otimes \sigma_1)_{A'}]/16$, where $T_{1,2} \neq 0$. Then Eq. (21) is violated, since the left hand side of Eq. (21) is proportional to $\tilde{i}T_1T_2(\sigma_3 \otimes \sigma_1) \otimes (\sigma_2 \otimes \sigma_0) \neq 0$ while the right hand side equals zero. Therefore, Theorem 2(a) does not hold for channels of higher dimensions. It means that high-dimension channels are easier to be super-activated.

Now we have studied the case where Λ_1 and Λ_2 are both single-qubit channels. We will briefly show that Theorem 2 does not hold for the general situation where Λ_1 and Λ_2 are qu$d$it channels with $d \geq 3$. A qu$d$it channel (with $d \geq 3$) has zero QCP if and only if it is either a completely decohering channel or an isotropic channel. It is obvious that, when Λ_2 is a completely depolarizing channel, Eq. (3) holds for any commutative states ξ_1 and ξ_2 which satisfy Eq. (6). However, for Case (a) in Theorem 2 where both Λ_1 and Λ_2 are identical isotropic channels, Eq. (3) can be violated by some commutative states ξ_1 and ξ_2 which satisfy Eq. (6). For example, consider A and A' are both

III. SUPER-ADDITIVITY OF QCP FOR GENERAL CHANNELS

Here we will prove the super-additivity of QCP for general channels. In the following, we only discuss the problem in the regime that the quantum correlation Q in Eq. (1) is quantum discord, which is defined as

$$\delta_{B|A}(\rho) = \min_{\{F_i^A\}} S(\rho_{B|\{F_i^A\}}) - S_{B|A}(\rho), \tag{22}$$

where $S_{B|A}(\rho) = S(\rho) - S(\rho_A)$ with $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is conditional entropy, $\{F_i^A\}$ is a positive operator-valued measure (POVM) on qudit A, $S(\rho_{B|\{F_i^A\}}) = \sum_i p_i S(\rho_{B|F_i^A})$ with $p_i = \mathrm{Tr}(\rho F_i^A)$ and $\rho_{B|F_i^A} = \mathrm{Tr}_A(\rho F_i^A)/p_i$ is the average entropy of B after the measurement.

Theorem 3. When two channels Λ_1 and Λ_2 are used paralleled, the QCP of the composite channel $\Lambda_1 \otimes \Lambda_2$ is no less than the sum of the QCP for the two channels

$$\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) \geq \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2). \tag{23}$$

Proof. Let ρ_1 and ρ_2 be the optimal input state of Λ_1 and Λ_2 respectively, then we have $\mathcal{Q}(\Lambda_i) = D(\rho_i')$ with $\rho_i' \equiv \Lambda_i \otimes I(\rho_i)$, $i = 1, 2$. As proved in Ref. [17], the classical correlation is additive for separable states $J(\rho \otimes \sigma) = J(\rho) + J(\sigma)$ when ρ is a separable state. Obviously, ρ_i' are separable states. Therefore, we have

$$\delta(\rho_1' \otimes \rho_2') = \delta(\rho_1') + \delta(\rho_2'). \tag{24}$$

Since $\rho_1 \otimes \rho_2$ may not be the optimal input state for channel $\Lambda_1 \otimes \Lambda_2$, from the definition of QCP, $\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) \geq \delta(\rho_1' \otimes \rho_2') = \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2)$. This completes the proof.

From the discussions in the last section, we observe that $\mathcal{Q}(\Lambda_1 \otimes \Lambda_2) > \mathcal{Q}(\Lambda_1) + \mathcal{Q}(\Lambda_2)$ is quite common. Therefore, we ask the following question: are there situations where the QCP is additive for channels with positive QCP? We give a positive answer to this question by providing a class of channels whose QCP is additive.

Here we define measuring-and-preparing (MP) channel as the operation which measure on a fixed orthogonal basis and then prepare the qubit to predefined states conditioned on the measurement results. All MP channels are unitarily equivalent to

$$\Lambda^{\mathrm{MP}}(\rho) = \sum_{i=0}^{d-1} \langle i|\rho|i\rangle \eta_i, \tag{25}$$

where η_i are quantum states. Belonging to the set of MP channels are the completely decohering channels, as well as the single-qubit channel with maximum QCP, whose Kraus operators are $E_0^{\mathrm{M}} = |0\rangle\langle 0|$ and $E_1^{\mathrm{M}} = |+\rangle\langle 1|$. For any MP channels in form of Eq. (25), the optimal input state to reach the maximum quantum discord in the output state is

$$\rho^{\mathrm{MP}} = \sum_{i=0}^{d-1} p_i |i\rangle\langle i| \otimes |i\rangle\langle i|. \tag{26}$$

The reason is as follows. Writing the general form of an optimal input state $\rho = \sum_{i=0}^{d-1} q_i|\phi_i\rangle\langle\phi_i| \otimes |i\rangle\langle i|$, we have the corresponding output state

$$\rho' = \sum_{i=0}^{d-1} p_i \eta_i \otimes \rho_i, \tag{27}$$

where $p_i = \sum_j q_j|\langle i|\phi_j\rangle|^2$ and $\rho_i = \sum_j q_j|\langle i|\phi_j\rangle|^2|j\rangle\langle j|/p_i$. Eq. (27) can be obtained from $\Lambda^{\mathrm{MP}} \otimes I(\rho^{\mathrm{MP}})$ by local operation on B, which can not increase the quantum discord. Therefore, the optimal input state should be in form of Eq. (26).

Theorem 4. When a MP channel Λ^{MP} and a completely decohering channel Λ^{CD} are used paralleled, the QCP of the composed channel equals to that of Λ^{MP}

$$\mathcal{Q}(\Lambda^{\mathrm{MP}} \otimes \Lambda^{\mathrm{CD}}) = \mathcal{Q}(\Lambda^{\mathrm{MP}}). \tag{28}$$

Proof. Consider the general form of an optimal input state

$$\rho = \sum_{i,j=0}^{d-1} p_{ij} \Pi_{\phi_{ij}}^{AA'} \otimes \Pi_{\psi_{ij}}^{BB'}, \tag{29}$$

where $|\phi_{ij}\rangle = U|ij\rangle$ and $|\psi_{ij}\rangle = |ij\rangle$. Without loss of generality, we assume that the basis of the completely decohering channel is $\{|i\rangle\}$ and that the MP channel is in the form of Eq. (25). The QCP of the channel Λ^{MP} is just the quantum discord in state $\Lambda^{\mathrm{MP}} \otimes I(\rho^{\mathrm{MP}})$. Notice that

$$\Lambda_{A'}^{CD}(\Pi_{\phi_{ij}}^{AA'}) = \sum_{k=0}^{d-1} q_{ij}^k \rho_{ij}^{(k)A} \otimes |k\rangle_{A'}\langle k|, \tag{30}$$

the state $\rho' = \Lambda_{A'}^{CD}(\rho)$ is of form

$$\rho' = \sum_{k=0}^{d-1} |k\rangle_{A'}\langle k| \otimes \sum_{i,j=0}^{1} p_{ij} q_{ij}^k \rho_{ij}^{(k)A} \otimes \Pi_{\beta_{ij}}^{BB'}. \tag{31}$$

The output state $\tilde{\rho} = \Lambda_A^{MP} \otimes \Lambda_{A'}^{CD}(\rho)$ is then

$$\tilde{\rho} = \sum_{k=0}^{d-1} r_k |k\rangle_{A'}\langle k| \otimes \rho_k. \tag{32}$$

where $r_k = \sum_{i,j=0}^{d-1} p_{ij} q_{ij}^k$, $\rho_k = \sum_l \eta_l^A \otimes \xi_{lk}^{BB'}$ and $\xi_{lk}^{BB'} = \sum_{i,j=0}^{1} p_{ij} q_{ij}^l \langle k|\rho_{ij}^{(l)}|k\rangle \Pi_{\beta_{ij}}^{BB'}/r_k$. Obviously, $\delta_{BB'|A}(\rho_k) \leq \mathcal{Q}(\Lambda^{MP})$ for $k = 1, 2, \cdots, d-1$. We will prove in the following that for states in the form of Eq. (32), the quantum discord of $\tilde{\rho}$ is lower bounded by the weighted average quantum discord of ρ_k

$$\delta_{BB'|AA'}(\tilde{\rho}) \leq \sum_{k=0}^{d-1} r_k \delta_{BB'|A}(\rho_k). \tag{33}$$

Suppose the optimal POVM for ρ_k are $\{F_k^{(i)}\}_{i=0}^{N_k-1}$ respectively. By building a POVM on qubits A and A' as $\{G^{(l)}\}_{l=0}^{\sum_k N_k-1} = \{|k\rangle\langle k| \otimes F_k^{(i)}\}_{k=0,\cdots,d-1}^{i=0,\cdots,N_k-1}$, we have

$$\begin{aligned} S(\tilde{\rho}_{BB'|\{F_{AA'}^{(l)}\}}) &= \sum_l \tilde{p}_l S(\frac{\mathrm{Tr}_{AA'}(G_{AA'}^{(l)}\tilde{\rho})}{\tilde{p}_l}) \\ &= \sum_{ik} r_k \tilde{q}_{ki} S(\frac{\mathrm{Tr}_{A'}(F_{k,A'}^{(i)}\rho_k)}{\tilde{q}_{ki}}) \\ &= \sum_{k=0}^{d-1} r_k S(\rho_{k,BB'|\{F_k^{(i)}\}}). \end{aligned} \tag{34}$$

Meanwhile, direct calculation leads to $S(\tilde{\rho}_{BB'|AA'}) = \sum_{k=0}^{d-1} r_k S(\rho_{k,BB'|AA'})$. Consequently, $S(\tilde{\rho}_{BB'|\{G_{AA'}^{(l)}\}}) - S(\tilde{\rho}_{BB'|AA'}) = \sum_{k=0}^{d-1} r_k \delta_{BB'|A}(\rho_k)$. By noticing that $\{G^{(l)}\}$ may not be the optimal POVM for the quantum discord of $\tilde{\rho}$, we have proved Eq. (33). Since Eq. (29) is a general form of optimal input state, the above discussion shows that

$$\mathcal{Q}(\Lambda^{\mathrm{MP}} \otimes \Lambda^{\mathrm{CD}}) \leq \mathcal{Q}(\Lambda^{\mathrm{MP}}). \tag{35}$$

Combining Eq. (35) with theorem 3, we finally reach Eq. (28). This completes the proof of theorem 4.

It should be noticed that both MP channel and CD channel are coherence-breaking channels. A CD channel takes any state to a state which is diagonal on a fixed basis, and thus the coherence between different state basis is broken. For a MP channel, coherence is broken during the measurement, and so does the genuine quantum correlation. Even through the preparation process in the MP channel can rebuild the bipartite quantum correlation, the genuine quantum correlation, which enable the super-additivity of QCP, can not be rebuilt. This is the reason why the QCP of a MP channel and a CD channel is additive. Therefore, we conjecture that the QCP of channels which are neither MP channels nor CD channels are super-additive.

IV. CONCLUSION

We have investigated the effect that, when two local channels are used paralleled, the QCP of the composite channel can be greater than the sum of QCP of the two channels. Two zero-QCP channels have the property of super-activation of QCP except the trivial cases that the two channels are both completely deochering channels or unitary channels. This result shows that super-activation of QCP is a fairly common effect for local channels. We also prove the super-additivity of QCP for general local channels and find a class of quantum channels whose QCP is additive.

Super-additivity of QCP is a collective effect. The genuine quantum correlation is observed in the initial state which can detect the super-activation of QCP for a CD

channel paralleled with a depolarizing channel. Meanwhile, the QCP of a MP channel and a CD channel is additive since both of them are coherence-breaking channels, which break the genuine quantum correlation. Therefore, we conjecture that genuine quantum correlation is responsible for such effect. This provide a new perspective to look at the concept of genuine quantum correlation, which is still an open problem in quantum information theory. From this point of view, our study can shed light on both classification of quantum channels and the structure of quantum correlation is multipartite states.

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